A) BASIC ELEMENTS.

A.1) The displacement vector. Lagrangean (material) and Eulerian (spatial) co-ordinates.

Consider an arbitrary material point inside a continuum body, subject to a deformation process. At the initial time t_0, that point has the position vector denoted by $\vec{X}$, with respect to the origin of a co-ordinate system (Fig.A1). At a time $t \geq t_0$, the new position vector is to be $\vec{x}$. The difference $\vec{x} - \vec{X}$ represents **the displacement vector**. Taking into account that the components (x_1, x_2, x_3) of the vector $\vec{x}$ are all functions of the components (X_1, X_2, X_3) of $\vec{X}$, the displacement vector can be written as

$$\vec{U} = \vec{U}(X_1, X_2, X_3, t) \quad . \tag{a1}$$

This represents a **Lagrangean (material)** description of the deformation process. Here, (X_1, X_2, X_3) are representing the **Lagrangean (material)** co-ordinates. Alternately, the components (X_1, X_2, X_3) of $\vec{X}$ can be seen as functions of the components (x_1, x_2, x_3) of the vector $\vec{x}$. Consequently, the displacement vector can be written as

$$\vec{u} = \vec{u}(x_1, x_2, x_3, t) \quad . \tag{a2}$$

This represents a **Eulerian (spatial)** description of the deformation process, where (x_1, x_2, x_3) are the **Eulerian (spatial)** co-ordinates. A basic supposition assumed thoroughly in that notes is that the deformation process is a continuous one, i.e. all the components of $\vec{u}$ or $\vec{U}$ are continuous functions together their derivatives with respect to both their spatial co-ordinates or to time.

The Lagrangean co-ordinates are usual in the Solid Mechanics, while the Eulerian co-ordinates are commonly used in Fluid Mechanics. However, in the Linear Elasticity, the distinction between these two kinds of co-ordinates is not important, as it will be seen in the next chapters. More details on such aspects can be found in (Aki and Richards 1980; Ranalli 1987).

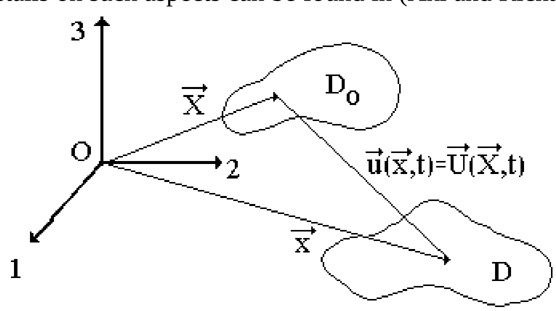

Fig.A.1. The continuum deformed body and the displacement vector.

A.2) Invariants of a tensor. Tensor deviator.

A second order tensor represents mainly a 3x3 matrix. The elements of the tensor are changing according to a certain rule with respect to a change of the co-ordinate system. Such a change with respect to a rotation will be discussed later. For simplicity, only symmetric tensors will be considered. A symmetric tensor is equal to its transpose

$$T = T^t \tag{a3}$$

(or $T_{ij} = T_{ji}$). The superscript “t” shows the transposed tensor (matrix).

Let the components of the tensor be real numbers

$$\mathrm{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} , \qquad \text{(a4)}$$

The scalar λ and the vector $\vec{u}$ are representing the eigen-value and the eigen-vector respectively of that tensor if

$$\mathrm{T}\,\vec{u} = \lambda\,\vec{u} \quad , \quad \vec{u} \neq \vec{0} \quad . \qquad \text{(a5)}$$

It’s easy to see that the eigen-values are not changing with respect to a rotation of the co-ordinates system.

Suppose now that the eigen-value λ and the components of the eigen-vector $\vec{u}$ are complex numbers. By taking the complex conjugate (denoted by an asterisk) into (a5), it follows

$$\mathrm{T}\,\vec{u}^{*} = \lambda\,\vec{u}^{*} \qquad \text{(a6)}$$

Taking into account the symmetry of the tensor, the next inner product is evaluated into two different ways

$$< \mathrm{T}\,\vec{u}, \vec{u}^{*} > = < \lambda\,\vec{u}, \vec{u}^{*} > = \lambda \left| \vec{u} \right|^{2} , \qquad \text{(a7)}$$

and

$$< \mathrm{T}\,\vec{u}, \vec{u}^{*} > = < \vec{u}, \mathrm{T}^{t}\vec{u}^{*} > = < \vec{u}, \mathrm{T}\vec{u}^{*} > = < \vec{u}, \lambda^{*}\vec{u}^{*} > = \lambda^{*} \left| \vec{u} \right|^{2} \qquad \text{(a8)}$$

From (a7) and (a8) it follows that the eigen-values (and the components of the eigen-vectors) of a symmetric tensor are real numbers.

Eq.(a5) can be written as

$$\left(\mathrm{T} - \lambda \mathbf{1} \right) \vec{u} = \vec{0} \quad , \quad \vec{u} \neq \vec{0} \quad , \qquad \text{(a9)}$$

where $\mathbf{1}$ denotes the unit tensor. From (a9) it follows that the next determinant vanishes

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{12} & T_{22} - \lambda & T_{23} \\ T_{13} & T_{23} & T_{33} - \lambda \end{vmatrix} = 0 \qquad \text{(a10)}$$

Hence the eigen-values are the roots of the third degree equation

$$-\lambda^{3} + I_{1}\lambda^{2} - I_{2}\lambda + I_{3} = 0 \quad , \qquad \text{(a11)}$$

where

$$I_{1} = T_{11} + T_{22} + T_{33} = \mathrm{tr}(\mathrm{T}) \quad ,$$
$$I_{2} = T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11} - T_{12}^{2} - T_{23}^{2} - T_{13}^{2} \quad , \qquad \text{(a12)}$$
$$I_{3} = T_{11}T_{22}T_{33} + T_{12}T_{23}T_{13} + \ldots = \det(\mathrm{T})$$

With respect to a rotation of the co-ordinates system, the elements of the tensor are generally changing. Because the quantities defined by (a12) can also be expressed as functions of the roots of (a11), it follows their values are not changing with respect to a rotation. They represent the main invariants of the tensor. The first invariant is **the trace of the tensor**, while the third one represent just its determinant.

The tensor defined by

$$\mathrm{T}^{*} = \mathrm{T} - \frac{1}{3}\mathrm{tr}(\mathrm{T})\mathbf{1} \quad , \qquad \text{(a13)}$$

represents the tensor deviator, having its trace equal to zero. Elementary computations show its second invariant is

$$I_2^* = -\frac{1}{6}\left[(T_{11}-T_{22})^2 + (T_{22}-T_{33})^2 + (T_{33}-T_{11})^2 + 6\left(T_{12}^2 + T_{23}^2 + T_{13}^2\right)\right] \qquad \text{(a14)}$$

That invariant is especially important to define constitutive equations for plasticity.

A.3) Strain tensor. Stress tensor. Equation of motion / equilibrium.

By using the spatial co-ordinates, the **strain tensor** is defined as

$$\varepsilon = \frac{1}{2}\left(\text{grad}\ \vec{u} + \text{grad}^t\ \vec{u}\right) \ . \qquad \text{(a15)}$$

where "grad" denotes the gradient. In Cartesian co-ordinates, that symmetric tensor has the elements

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial uj}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \ . \qquad \text{(a16)}$$

According to the CAUCHY's hypotheses there are two kinds of forces acting at an arbitrary point placed inside a body or on its boundary. The first ones are represented by **the mass forces**, characterised by a mass density $\vec{b}$. For the problems discussed in that book, such mass forces are ignored. Or, they are represented by the gravity, when $\vec{b}$ is just the gravitational acceleration $\vec{g}$. Suppose now a mechanical state of tension (stress) is present inside the deformed body, e.g. as a result of the action of a pair forces $\pm\vec{T}$. An arbitrary cross section is considered through a certain point of the body, dividing it into a part denoted by D_l at left and a pert D_r at the right respectively (Fig.A.2). A surface element dS is considered on the boundary of D_l, having the outer pointing normal vector denoted by $\vec{n}$. The material points of the boundary of D_r are acting on dS by an elementary force $\vec{df}$. It follows (e.g. Aki and Richards 1980; Ranalli 1987) that the next relation is valid

$$\frac{\vec{df}}{dS} = \sigma\,\vec{n} \ , \qquad \text{(a.17)}$$

where the tensor σ represents **the CAUCHY stress tensor**, spatial co-ordinates being used. According to the Principle of the Kinetic Momentum Balance, stress is a symmetric tensor. It can be shown too that the Principle of Impulse Balance leads to the next equation of motion /equilibrium

$$\text{div}\,\sigma + \rho\,\vec{b} = \rho\frac{d^2\vec{u}}{dt^2} \ , \qquad \text{(a18)}$$

That equation is valid at an arbitrary point inside the body , where ρ is the density and $\frac{d^2\vec{u}}{dt^2}$ represents the acceleration. By projecting eq.(a18) on the co-ordinates system axes, three scalar equations are obtained.

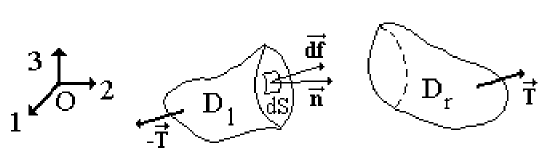

Fig.A.2. An imaginary cross section through the deformed body.

A.4) HOOKE's law.

Neglecting the initial stress (in most cases), it is further assumed a linear relation between the stress and strain tensors, i.e.

$$\sigma = H\,\varepsilon\ , \tag{a19}$$

or

$$\sigma_{ij} = H_{ijkl}\varepsilon_{kl}\ , \tag{a20}$$

where H is a fourth-order tensor. Eq.(a19) represents HOOKE's law. In the usual cases discussed here, an elastic, homogeneous, isotropic medium is considered. Then eq.(a19) takes the particular form

$$\sigma = \lambda \mathrm{tr}\,\varepsilon\, 1 + 2\mu\varepsilon\ . \tag{a21}$$

Here, tr denotes the trace of the tensor, 1 is the unit tensor(matrix) and λ, μ are the elastic coefficients of LAMÉ. Hence

$$\begin{aligned}
&\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{11},\\
&\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{22},\\
&\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{33}\\
&\sigma_{12} = \sigma_{21} = 2\mu\,\varepsilon_{12},\quad \sigma_{13} = \sigma_{31} = 2\mu\,\varepsilon_{13},\quad \sigma_{23} = \sigma_{32} = 2\mu\,\varepsilon_{23}
\end{aligned}\ , \tag{a.22}$$

Alternately, HOOKE's law (a21) can be reversed to give

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\mathrm{tr}\,\sigma\, 1\ , \tag{a23}$$

where the modulus of YOUNG is

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu} \tag{a24}$$

and the transverse contraction coefficient of POISSON is

$$\nu = \frac{\lambda}{2(\lambda + \mu)}\ . \tag{a25}$$

By reversing (a24) and (a25), it follows

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)}E\ ,\quad \mu = \frac{1}{2(1+\nu)}E\ . \tag{a26}$$

The parameter defined by

$$\chi = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)} \tag{a27}$$

represents the **incompressibility** or **bulk modulus**. For (theoretical) incompressible rocks, that modulus approaches infinity.

Other constitutive equations will be discussed in relation to the rheological bodies.